Characteristics of Strategy-Proof Fuzzy Choice*

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Abstract

This paper considers the strategic manipulation of fuzzy choice functions where both individuals and groups can choose alternatives to various degrees. Past efforts to model fuzzy choice and strategic manipulation have allowed individual preferences to be fuzzy but still required groups to select only one alternative (e.g. Abdelaziz, José and Meddeb [1]; Côrte-Real [14]). Under this new framework, I find, with very minimal assumptions on fuzzy preferences, strategy-proof fuzzy choice functions satisfy fuzzy versions of peak-only, weak Paretianism and monotonicity. In addition, the only type of strategy-proof fuzzy choice function corresponds to the traditional augmented median rule. Further, I illustrate the implications this framework in the spatial model. These results are relevant to the manipulation literature, which remains divided as to whether choice functions can be both non-manipulable and non-dictatorial when restricting individual preferences to a single-peaked domain (e.g. Mackie [24]; Penn, Patty and Gailmard [30]). In this context, the paper suggests that social choice can be both strategy-proof and non-dictatorial if alternatives are chosen to various degrees.

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1 Strategy-Proofness and Fuzzy Sets

A collective choice function over three or more alternatives that does not incentivize individuals who misrepresent their true preferences must be dictatorial [19, 34]. It then follows that voters in collective choice institutions will manipulate the voting procedure to obtain a more preferred social outcome by misrepresenting their true preferences. Social choice scholars have tried to avoid this conclusion of the Gibbard-Satterthwaite (G-S in what follows) theorem by relaxing several of its original assumptions. One approach restricts the domain of individual preferences to single-peaked profiles and finds that the augmented median rule emerges as a non-manipulable and non-dictatorial choice function [5, 9, 26, ?]. While some scholars (e.g. Dryzek and List [16]; Mackie [24]) hold that this restriction voids the results of the G-S theorem, Penn, Gailmard and Patty [30] extend the G-S results to a general case by demonstrating that even though individuals may possess single-peaked preferences, there exists opportunities to manipulate the social choice when individuals report false preferences that violate the natural ordering of the alternatives.

What has remained absent from this debate has been the effects of fuzzy preferences and fuzzy social choice on G-S's conclusion. In the fuzzy framework, individauls can prefer one alternative over another to a certain degree instead of only possessing strict preference or indifference between the two [4, 29]. The addition of fuzzy preferences then requires the specification of a fuzzy social choice function that selects some type of outcome. Although past efforts have explored situations where actors have fuzzy preferences but, as group, must make exact choices, i.e. where society selects one alternative unequivocally and have only confirmed the G-S conclusion [1, 14, 37], the strategic manipulation of truly fuzzy social choice functions, where society chooses alternatives to certain degrees, has yet to be considered.

The purpose of this paper is to address this lacuna in the manipulation literature. To do so, it integrates two recent developments in fuzzy social choice theory. First, it considers the specific fuzzy choice functions proposed by Dasgupta and Deb [15] and Banerjee [4]. These choice functions allow a set of individuals to select more than one alternative with varying degrees of choice. While there exist a significant literature investigating the manipulation of social choice correspondences that allow multiple alternatives to be selected (e.g. Kelly [22] and Barberà, Dutta and Sen [6]), the degree to which the group actors select the alternatives has not been allowed to vary. Second, the paper uses the fuzzy preference framework proposed by Nurmi [28], which employs preference functions instead of preference relations to describe individual preferences. A preference function, unlike a preference relation, accounts for an individual's preference over the set of alternatives instead of over the set of ordered pairs of alternatives. This approach not only allows for a direct comparison between individual preferences and the social choice, but it is also more conducive to empricial testing and application because it does not require an individual to immediately specify his or her preference comparing every alternative to another [8, 12, 13].

Under this setup, I characterize fuzzy choice functions and demonstrate that strategy-proofness implies fuzzy versions of weak Paretianism, peak-only and monotonicity. Further, strategy-proofness is necessary and sufficient for the augmented median voter rule. These results suggest that when the social choice is allowed to be fuzzy, there exists a class of non-manipulable choice functions, which do not replace any type of transitivity or single-peaked restriction on individual preferences. In addition, I illustrate the model using fuzzy spatial preferences. The paper proceeds as follows. Section two briefly reviews the literature discussing fuzzy manipulation. Section three presents the main concepts and definition. Section four details the main findings of the paper. Finally, section five offers a discussion and a critique of the social choice model in context of the spatial model, and section 6 concludes the paper.

2 Previous Attempts at Fuzzy Choice

Fuzzy Choice and Manipulation

Most efforts incorporating fuzzy mathematics into social choice functions start with a fuzzy preference relation, which is a function $\rho : X \times X \to [0, 1]$ where X is the set of alternatives. In words, $\rho(x, y)$ refers to the degree to which x is at least as good as y. If $\rho(x, y) = 1$, then x is said to be definitely as least as good as y; if $\rho(x, y) = 0$, then x is said to be definitely not as least as good as y. When $\rho(x, y) \in (0, 1)$, the preference for x over y is said to be vague or ambiguous.[1] For a set of n actors,N, previous definitions of fuzy choice functions associates an n-tuple of fuzzy preference relations with one alternative in X [1, 14, 29, 37].

Because a fuzzy preference relation is not directly comparable to a subset of alternatives, scholars have considered various mechanisms to aggregate individual preference relations into a social choice. Intitial studies assumed that individuals possess fuzzy preferences but must make "crisp" individual choices over the set of alternatives, and the choice function associates a set of alternatives to these crisp choices [14, 29]. Such situations arise when actors, who possess fuzzy preference relations must vote, "yes" or "no" for an amendment or select only one candidate among many. Later research aggregates a collection of individual preference relations into a social preference relation and then associates an alternative with the fuzzy social preference relation [1]. The following example illustrates the difference between the two approaches.

Example. Let $X = \{a, b\}$ and $N = \{1, 2, 3\}$. Suppose $\rho_1(a, b) = .4, \rho_2(a, b) = .4, \text{and} \rho_3(a, b) = .9$. In words, ρ_i is the fuzzy individual preference relation associated with $i \in N$. Futhermore, suppose reciprocity in preferences, and accordingly, $\rho_i(b, a) = 1 - \rho_i(a, b)$.

Orlovsky Rule. The Orlovsky [29] rule demonstrates the first approach to fuzzy choice, where each actor must make a crisp decision. The Orlovsky rule – or a variation of it – is a function that maps a set of two alternatives into the set $\{0, 1\}$, formally, $IC_i : X \to \{0, 1\}$.

[1] For a more thorough review fuzzy preferences and how they relate to tradition preferences see Orlovsky [29], Dutta [18], Richardson [33] and Llamazares [23].

More specifically, let $x, y \in X$ and the Orlovsky rule be defined as follows:

$$IC_i(\rho_i)(x) = \begin{cases} 1 & \text{if } R_i(x,y) > R_i(y,x) \\ 0 & \text{else} \end{cases}$$

In words, i votes for, or chooses, x over y if and only if $IC_i(\rho_i)(x) = 1$ and $IC_i(\rho_i(y) = 0$. Considering the above example, $IC(\rho)(a) = (IC_1(\rho_1)(a), IC_2(\rho_2)(a), IC_3(\rho_3)(a)) = (0,\ 0,\ 1)$, $IC(\rho)(b) = (IC_1(\rho_1)(b), IC_2(\rho_2)(b), IC_3(\rho_3)(b)) = (1,\ 1,\ 0)$, because actors 1 and 2 choose b and actor 1 chooses a. We can tally the votes in any number of ways, but under majority rule b is the outcome.

Mean Aggregation Rule. The mean aggregation rule is a fuzzy aggregation rule and demonstrates the second type of fuzzy choice function, where actors need not make crisp decisions. The mean aggregation rule is defined as follows:

$$\rho_S(x,y) = \frac{1}{n}\sum_{i=1}^{n} \rho_i(x,y)$$

Using the mean aggregation rule, we can specify the social fuzzy preference relation, which is $\rho_S(a,b) = .567$ and $\rho_S(b,a) = .433$. When we use the Orlovsky rule on ρ_S, the social choice becomes a because $\rho_S(a,b) > \rho_S(b,a)$.

In both conceptualizations, the group only selects one, exact alternative even though the choice functions are said to be fuzzy. Further, they both return identical results to the G-S theorem where a choice function is non-manipulable if and only if it is dictatorial[1, 37]. Côrte-Real [14] demonstrates the Orlovsky rule is strategy proof but considers only two alternatives. Nonetheless, new results may be obtained when considering the fuzzy choice functions proposed by Dasgupta and Deb [15] and Banerjee [4]. Under their framework, choice is represented as a fuzzy subset of the set of alternatives, i.e. $\beta : X \to [0,1]$. For any alternative $x \in X$, $\beta(x)$ denotes the degree to which x is chosen. While the conceptualization of fuzzy choice has received a great deal of attentiaion in revealed preference theory[20, 21], the possiblity of manipulating these types of choice functions has yet to be considered.

Individual Preferences and Manipulation

Before presenting a model of fuzzy choice, a brief discussion on the types of fuzzy individual preferences is needed. Informally, a choice function is manipulable by an actor if the actor can unilaterally change the social choice in her favor by submitting an insincere or false preference. To address this formally, it follows that there exist some mechanism to compare the social choice with an individual's preferences. In the exact case, each individual possesses a transitive ranking of the alternatives and a choice is manipulable if there exists an individual who can move the social choice further up her ranking. In the fuzzy case, this mechanism relating individual preferences to the social choice is more complicated. When comparing individual fuzzy preference relations and an exact social choice, Abdelaziz, Figueira and Meddeb [1] utilize four different procedures determining whether an individual prefers one alternative over another, hence four definitions of manipulability. Perote-Peña and Piggins [31] offer one solution to the problem by considering the manipulation of fuzzy aggregation rules, where an n-tuple of fuzzy preference relations are aggregated into a single social preference relation. While this setup has only confirmed the G-S theorem in the fuzzy framework (see Duddy, Perote-Peña and Piggins [17] for a general proof), modeling individual preferences as fuzzy subsets of the set of alternatives rather than fuzzy relations simplifies the forthcoming analysis.

Further, representing individual preferences in this manner is not completely divoriced from preference relations. Dasgupta and Deb [15] and Georgecuscu [21] illustrate how fuzzy subsets can be related to fuzzy preference relations using concepts similar to R-maximality and R-greatness revealed preference theory (see Suzumura [36] and Sen [35] for reference). In addition, Clark, Larson, Mordeson, Potter and Wierman [11] discuss several substantive interpretations of fuzzy subsets of the set of alternatives as representations of individual preference. For example, let β be a fuzzy subset of X and $x \in X$. When $\beta(x)$ refers to the degree to which x is ideal, actors are uncertain how ideal each alternative is; however, the they are quite certain whether x is better, or preferred to, another alternative $y \in X$.

3 Fuzzifying Collective Choice

This section details the fuzzy choice framework and introduces the concepts of strategy-proofness used in the model. Let X be a set of alternatives either finite or infinite. A fuzzy subset of X, f, is a function $f: X \to [0,1]$. Let $\mathcal{F}(X)$ denote all possible fuzzy subsets of X. Let N be a finite set of individuals, where $N = \{1, 2, \ldots, n\}$ and $n \geq 2$.

Definition 1. (*Fuzzy preference function*). An individual fuzzy preference function on X associated with $i \in N$ is a function $\sigma_i : X \to [0,1]$.

Obviously, $\sigma_i \in \mathcal{F}(X)$. Suppose $x \in X$. Then the literature, $\sigma_i(x)$ provides two substantive interpretations of a fuzzy preference function. First, $\sigma_i(x)$ can refer to the degree to which $i \in N$ views x as ideal, where $\sigma_i(x) = 0$ means i believes x is abhorrent and $\sigma_i(x) = 1$ means i believes x is ideal [28, 11, **?**]. Second, $\sigma_i(x)$ can also be interpretted as the degree to which i chooses the alternative x [4, 20, 21]. In this paper, $\sigma_i(x)$ is often refered as the choice instensity of individual i for alternative x. The *profile* of all individual fuzzy preference functions can be written as $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$. $\sigma(x) = (\sigma_1(x), \sigma_2(x), \ldots, \sigma_n(x))$ denotes the restriction of σ to x. A fuzzy choice function then associates a fuzzy subet of X to a profile of individual fuzzy preference functions.

Definition 2. (*Fuzzy choice function*). A fuzzy choice function is a function $\mathcal{C}(\sigma) : \mathcal{F}(X)^n \to \mathcal{F}(X)$.

Let $\mathcal{C}(\sigma)(x)$ denote the degree to which the group of individuals, N, chooses $x \in X$ given a specific $\sigma \in \mathcal{F}(X)^n$ and fuzzy choice funtion $\mathcal{C}$. It is assumed that $\mathcal{C}(\sigma)$ has *full range*: for any $x \in X$, there exists a $\sigma \in \mathcal{F}(X)^n$ such that $C(\sigma)(x) = \alpha$, for all $\alpha \in [0,1]$.[2] Full range also guarantees that $\mathcal{C}$ is nontrivial, i.e. $\mathcal{C}(\sigma) \neq c$ for all $\sigma \in \mathcal{F}(X)^n$, where c is some constant in $[0,1]$. In addition, $(\sigma_{N\setminus i}, \sigma'_i)$ represents the profile of individual preference functions where $(\sigma_1, \sigma_2, \ldots, \sigma'_i, \ldots, \sigma_n)$ and $(\sigma_{N\setminus i}(x), \sigma_i(x))$ is the profile's restriction to $x \in X$.

[2] Some authors refer to the full range assumption as citizen sovereignty when preferences are required to be single-peaked [3].

The following definitions characterize several properties of fuzzy choice functions.

Definition 3. (*Weakly Paretian*). A fuzzy choice function $\mathcal{C}(\sigma)$ is said to be weakly paretian if for all $\sigma \in \mathcal{F}(X)^n$ and all $x \in X$,

$$\max_{i \in N}(\sigma_i(x)) \geq \mathcal{C}(\sigma)(x) \geq \min_{i \in N}(\sigma_i(x)).$$

In words, weak Paretianism guarrantees that the degree to which a fuzzy choice function selects an alternative is (1) not greater than the choice intensity of the individual who chooses the alternative to the most intense degree and (2) not less than the choice intensity of the individual who chooses the alternative to the least intense degree.

Definition 4. (σ-only). A fuzzy choice function $\mathcal{C}$ is said to satisfy the σ-only condition if for all σ, $\sigma' \in \mathcal{F}(X)^n$ and all $x \in X$ such that $\sigma_i(x) = \sigma'_i(x)$ for all $i \in N$,

$$\mathcal{C}(\sigma)(x) = \mathcal{C}(\sigma')(x).$$

In words, the σ-only condition guarrantees that the degree to which a fuzzy choice function for all $x \in X$ is independent of the choice intensities assigned to other alternatives.

Definition 5. (*Monotonic*). A fuzzy choice function $\mathcal{C}$ is said to be monotonic if, for all $x \in X$, all $\sigma \in \mathcal{F}(X)^n$, and all $\sigma'_i \in \mathcal{F}(X)$,

$$\sigma_i(x) \leq \sigma'_i(x), \forall i \in N \implies \mathcal{C}(\sigma)(x) \leq \mathcal{C}(\sigma_{N \setminus i}, \sigma'_i)(x)$$

Monotonicity requires that increasing the degree to which individuals choose a specific alternative will not decrease the degree of social choice for that alternative.

Definition 6. (*Manipulable*). A fuzzy choice function $\mathcal{C}$ is manipulable if there exixts $x \in X$, $\sigma \in \mathcal{F}(X)^n$, $i \in N$ and $\sigma'_i \in \mathcal{F}(X)$ such that

$$(1)\ \ \mathcal{C}(\sigma)(x) < \sigma_i(x) \Rightarrow \mathcal{C}(\sigma)(x) < \mathcal{C}(\sigma_{N\setminus i},\, \sigma_i')(x)$$

$$(2)\ \ \mathcal{C}(\sigma)(x) > \sigma_i(x) \Rightarrow \mathcal{C}(\sigma)(x) > \mathcal{C}(\sigma_{N\setminus i},\, \sigma_i')(x)$$

According to definition 6, a fuzzy choice function is manipulable if an individual $i \in N$ is able to move the degree of social choice for an alternative in the direction of her sincere choice intesnity of that alternative by submitting a false preference for x.

Definition 7. (*Strategy-Proof (SP)*). A fuzzy choice function $\mathcal{C}$ is said to be strategy-proof if it is not manipulable.

Example. Let $\sigma \in \mathcal{F}(X)^n$ and let $\mathcal{C}$ be a fuzzy choice function. Suppose for some $x \in X$ there exists an $i \in N$ such that $\sigma_i(x) = .4$. Suppose $\mathcal{C}(\sigma)(x) = .3$ and for all $\sigma_i' \in \mathcal{F}(X)$, $\mathcal{C}(\sigma_{N\setminus i}, \sigma_i')(x) = .9$. Hence, $\mathcal{C}$ is manipulable by definition 6 even though $|\sigma_i(x) - \mathcal{C}(\sigma)(x)| < |\sigma_i(x) - \mathcal{C}(\sigma_{N\setminus i}, \sigma_i')(x)|$.

The preceeding example begs the question is a fuzzy choice function manipulable even when it over corrects, of sorts, for manipulation? More specifically, it illustrates a situation where, given Euclidean preferences over the range of the choice function, a fuzzy choice function is manipulable even though the choice function produces an output further away from i's ideal choice instensity when manipulated. Because of this, some authors define SP in the following context [31].

Definition 8. (*Strategy-Proof'*) A fuzzy choice function $\mathcal{C}$ is strategy-proof if for all $x \in X$, all $\sigma \in \mathcal{F}(X)^n$ and all $i \in N$ the following hold for all $\sigma_i' \in \mathcal{F}(X)$:

$$(1)\ \ \mathcal{C}(\sigma)(x) < \sigma_i(x) \Rightarrow \mathcal{C}(\sigma)(x) \geq \mathcal{C}(\sigma_{N\setminus i},\, \sigma_i')(x)$$

$$(2)\ \ \mathcal{C}(\sigma)(x) > \sigma_i(x) \Rightarrow \mathcal{C}(\sigma)(x) \leq \mathcal{C}(\sigma_{N\setminus i},\, \sigma_i')(x)$$

In words, definition 9 guarrantees that when an individual i submits a false preference profile, σ_i, the resulting social choice is "at least as far away" from the original social choice when every individual submits his or her true preference. Hence, individual i has no incentive manipulate the social choice. However, in a formal context, definition 9 and definition 8 are equivalent.

Proposition 9. *Strategy-poofness under definition 8 and strategy-proofness under definition 9 are equivalent.*

Proof. Def. 8 $\implies$ Def. 9. Let $\mathcal{C}$ be a choice function that is strategy-proof under definition 8. Let $\sigma \in \mathcal{F}(X)^n$, $x \in X$ and $i \in N$ such that $\mathcal{C}(\sigma)(x) < \sigma_i(x)$. Because $\mathcal{C}$ is strategy-proof under definition 8, there does not exist $\sigma'_i \in \mathcal{F}(X)$ such that $\mathcal{C}(\sigma)(x) < \mathcal{C}(\sigma_{N\setminus i}, \sigma'_i)(x)$. Thus, for all $\sigma'_i \in \mathcal{F}(X)$, $\mathcal{C}(\sigma)(x) \geq \mathcal{C}(\sigma_{N\setminus i}, \sigma'_i)(x)$. An symmetrical argument can be made for $\mathcal{C}(\sigma)(x) > \sigma_i(x)$. Because $\sigma \in \mathcal{F}(X)^n$, $x \in X$ and $i \in N$ are arbitrary, strategy-poofness under definition 8 implies strategy-proofness under definition 9.

Def. 9 $\implies$ Def. 8. Let $\mathcal{C}$ be a choice function that is strategy-proof under definition 9. Let $\sigma \in \mathcal{F}(X)^n$, $x \in X$ and $i \in N$ such that $\mathcal{C}(\sigma)(x) < \sigma_i(x)$. Because $\mathcal{C}$ is strategy-proof under definition 9, for all $\sigma'_i \in \mathcal{F}(X)$, $\mathcal{C}(\sigma)(x) \geq \mathcal{C}(\sigma_{N\setminus i}, \sigma'_i)(x)$. Then there does not exist a $\sigma'_i \in \mathcal{F}(X)$ such that $\mathcal{C}(\sigma)(x) < \mathcal{C}(\sigma_{N\setminus i}, \sigma'_i)(x)$. A symmetrical argument holds for $\mathcal{C}(\sigma)(x) > \sigma_i(x)$. Thus, $\mathcal{C}$ is strategy-proof under definition 8. □

4 Findings

This section details the main findings of the paper. To characterize the properties of fuzzy choice functions, the following formal arguments utilize several reinterpretations of the pivotal voter theorem presented in Reny [32].

Proposition 10. *If a fuzzy choice function $\mathcal{C}$ is strategy-proof, then it satisfies the σ-only condition.*

Proof. Assume $\mathcal{C}$ is SP. Now suppose $\mathcal{C}$ is not σ-only. This proof will show that this leads to a contradiction. By the assumption, there exists σ, $\sigma' \in \mathcal{F}(X)^n$ and an $x \in X$ such that $\sigma(x) = \sigma'(x)$ and $\mathcal{C}(\sigma)(x) \neq \mathcal{C}(\sigma')(x)$.

There are two cases for consideration. Now construct the following profile of individual preferences $Z = \{z_0,\ z_1, ...,\ z_k, ...,\ z_n\}$ such that the following hold:

$$
\begin{aligned}
z_0 &= (\sigma_1, ...,\ \sigma_i, ...,\ \sigma_n) \\
z_1 &= (\sigma_1', ...,\ \sigma_i, ...,\ \sigma_n) \\
&\vdots \\
z_i &= (\sigma_1', ...,\ \sigma_i', ...,\ \sigma_n) \\
&\vdots \\
z_n &= (\sigma_1', ...,\ \sigma_i', ...,\ \sigma_n'),
\end{aligned}
$$

where $z_{i,j}$ signifies $\sigma_j \in z_i$ and $z_{i\backslash i}$ denotes $z_i \backslash \{\sigma_i\}$. It is apparent that there exist some $z_{i-1}, z_i \in Z$ such that $\mathcal{C}(z_{i-1})(x) = \mathcal{C}(\sigma)(x)$ and $\mathcal{C}(z_i)(x) \neq \mathcal{C}(\sigma)(x)$. Without loss of generality, suppose this occurs at $i = 2$. This proof will show that $i = 2$ can manipulate $\mathcal{C}$ at z_1 and z_2. Now there are two cases to consider.

Case 1. $\mathcal{C}(z_1)(x) < \mathcal{C}(z_2)(x)$. First, suppose $\sigma_2(x) < \mathcal{C}(z_2)(x)$. Here, $\mathcal{C}(z_2)(x) > \mathcal{C}(z_{2\backslash 2}, z_{1,2}) = \mathcal{C}(z_1)(x)$. Thus, $i = 2$ can manipulate $\mathcal{C}$ at z_2 by submitting $z_{1,2}$ rather than $z_{2,2}$. Second, suppose $\sigma_2(x) \geq \mathcal{C}(z_2)(x)$. Because $\mathcal{C}(z_1)(x) < \mathcal{C}(z_2)(x)$, then $\sigma_2(x) > \mathcal{C}(z_1)(x)$. However, $\mathcal{C}(z_1)(x) < \mathcal{C}(z_{1\backslash 2}, z_{2,2}) = \mathcal{C}(z_2)(x)$. Thus, $i = 2$ can manipulate $\mathcal{C}$ at z_1 by submitting $z_{2,2}$ rather than $z_{1,2}$. This is a contradiction. Hence, $\mathcal{C}(z_1)(x) \geq \mathcal{C}(z_2)(x)$.

Case 2. $\mathcal{C}(z_1)(x) > \mathcal{C}(z_2)(x)$. First, suppose $\sigma_2(x) < \mathcal{C}(z_1)(x)$. Here, $\mathcal{C}(z_1)(x) > \mathcal{C}(z_{1\backslash 2}.z_{2,2}) = \mathcal{C}(z_2)(x)$. Thus, $i = 2$ can manipulate $\mathcal{C}$ at z_1 by submitting $z_{2,2}$ rather than $z_{1,2}$. Second, suppose $\sigma_2(x) \geq \mathcal{C}(z_1)(x)$. Because $\mathcal{C}(z_1)(x) > \mathcal{C}(z_2)(x)$, then $\sigma_2(x) > \mathcal{C}(z_2)(x)$. However, $\mathcal{C}(z_2)(x) < \mathcal{C}(z_{2\backslash 2}, z_{1,2})(x) = \mathcal{C}(z_2)(x)$, which is another contradiction. Hence, $\mathcal{C}(z_1)(x) = \mathcal{C}(z_2)(x)$.

The above argument can be replicated for any $i \in N$. Thus, for any $z_i, z_j \in Z$, $\mathcal{C}(z_i)(x) = \mathcal{C}(z_j)(x)$. Accordingly, $\mathcal{C}(\sigma)(x) = \mathcal{C}(z_0)(x) = \mathcal{C}(z_n)(x) = \mathcal{C}(\sigma')(x)$. Hence, $\mathcal{C}$ satisfies the σ-only condition. □

Proposition 11. *If a fuzzy choice function $\mathcal{C}$ is strategy-proof, then it is weakly Paretian.*

Proof. Assume $\mathcal{C}$ is strategy-proof. Now, suppose $\mathcal{C}$ is not weakly Paretian. This proof will show that this leads to a contradiction.There are two cases for consideration.

Case 1. Suppose there exists $x \in X$ and $\sigma \in \mathcal{F}(X)^n$ such that $\mathcal{C}(\sigma)(x) < \min_{i \in N}(\sigma_i(x))$. By full range, we know there also exists a $\sigma' \in \mathcal{F}(X)^n$ such that $\mathcal{C}(\sigma')(x) \geq \min_{i \in N}(\sigma'_i(x))$. Now construct a vector of profiles $Z = (z_0,\ z_1, ...,\ z_i, ...,\ z_n)$ such that

$$
\begin{aligned}
z_0 &= (\sigma_1(x), ...,\ \sigma_i(x), ...,\ \sigma_n(x)) \\
z_1 &= (\sigma'_1(x), ...,\ \sigma_i(x), ...,\ \sigma_n(x)) \\
&\vdots \\
z_i &= (\sigma'_1(x), ...,\ \sigma'_i(x), ...,\ \sigma_n(x)) \\
&\vdots \\
z_n &= (\sigma'_1(x), ...,\ \sigma'_i(x), ...,\ \sigma'_n(x)),
\end{aligned}
$$

where $z_{i,j}$ signifies $\sigma_j \in z_i$ and $z_{i \setminus i}$ denotes $z_i \setminus \{\sigma_i\}$. It is obvious that there exists $z_{i-1}, z_i \in Z$ such that $\mathcal{C}(z_{i-1})(x) < \mathcal{C}(z_i)(x)$. Suppose $\sigma_i(x) > \mathcal{C}(z_{i-1})(x)$. Then $\mathcal{C}(z_{i-1})(x) < \mathcal{C}(z_{i-1\setminus i}, z_{i,i})(x) = \mathcal{C}(z_i)(x)$. Thus, i can manipulation $\mathcal{C}$ at (z_{i-1}) by submitting $z_{i,i}$ rather than $z_{i-1,i}$. Now suppose $\sigma_i(x) \leq \mathcal{C}(z_{i-1})(x)$, then $\sigma_i(x) < \mathcal{C}(z_i)(x)$. However, $\mathcal{C}(z_i)(x) > \mathcal{C}(z_{i\setminus i}, z_{i-1,i})(x) = \mathcal{C}(z_{i-1})(x)$. Thus, i can manipulate $\mathcal{C}$ at (z_i) by submitting $z_{i-1,i}$ rather than $z_{i,i}$. This is a contradiction.

Case 2. Suppose there exists $x \in X$ and $\sigma \in \mathcal{F}(X)^n$ such that $\mathcal{C}(\sigma)(x) > \max_{i \in N}(\sigma_i(x))$. By full range, we know there also exists a $\sigma' \in \mathcal{F}(X)^n$ such that $\mathcal{C}(\sigma')(x) \leq \max_{i \in N}(\sigma'_i(x))$. Now construct a vector of profiles $Z = (z_0,\ z_1, ...,\ z_i, ...,\ z_n)$ in a manner detailed above. Likewise, there exists $z_{i-1}, z_i \in Z$ such that $\mathcal{C}(z_{i-1})(x) > \mathcal{C}(z_i)(x)$. Suppose $\sigma_i(x) < \mathcal{C}(z_{i-1})(x)$. Then $\mathcal{C}(z_{i-1})(x) > \mathcal{C}(z_{i-1\backslash i}, z_{i,i}) = \mathcal{C}(z_i)(x)$. Thus, i can manipulate $\mathcal{C}$ at (z_{i-1}) by submitting $z_{i,i}$ rather than $z_{i-1,i}$. Now suppose $\sigma_i(x) \geq \mathcal{C}(z_{i-1})(x)$, then $\sigma_i(x) > \mathcal{C}(z_i)(x)$. However, $\mathcal{C}(z_i)(x) < \mathcal{C}(z_{i\backslash i}, z_{i-1,i})(x) = \mathcal{C}(z_{i-1})(x)$, another contradiction.

Hence, $\max\limits_{i \in N}(\sigma_i(x)) \geq \mathcal{C}(\sigma)(x) \geq \min\limits_{i \in N}(\sigma_i(x))$, $\mathcal{C}$ is weakly Paretian. □

Corollary 12. *If a fuzzy choice function $\mathcal{C}$ is strategy-proof but does not satisfy full range then it does not satisfy weak Paretianism.*

Proof. An example will suffice. Suppose, for all $x \in X$ and all $\sigma \in \mathcal{F}(X)^n$, $\mathcal{C}(\sigma)(x) = c$. In this case, $\mathcal{C}$ is strategy-proof but is not weakly Paretian if $c < \sigma_i(x)$ for all $i \in N$. □

Together, proposition 10 and corollary 11 demonstrate the equivalence of weak Paretianism and full range under strategy-proof choice functions. The next proposition and its subsequent corollary highlight the relationship between strategy-proofness and monotonicity.

Proposition 13. *If a fuzzy choice function $\mathcal{C}$ is strategy-proof, then it is monotonic.*

Proof. Assume $\mathcal{C}$ is SP. Now suppose $\mathcal{C}$ is not monotonic. This proof will illustrate that this leads to a contradiction. Because $\mathcal{C}$ is not monotonic, there exists an $x \in X$ and σ, $\sigma' \in \mathcal{F}(X)^n$ such that $\sigma_i(x) \leq \sigma'_i(x)$, for all $i \in N$, and $\mathcal{C}(\sigma)(x) > \mathcal{C}(\sigma')(x)$. Because $\sigma \neq \sigma'$, there exists at least one $i \in N$ such that $\sigma_i(x) < \sigma'_i(x)$. Now construct the following vector of profiles $Z = (z_0,\ z_1, ...,\ z_i, ...,\ z_n)$ such that

$$
\begin{aligned}
z_0 &= (\sigma_1(x), ..., \sigma_i(x), ..., \sigma_n(x)) \\
z_1 &= (\sigma_1'(x), ..., \sigma_i(x), ..., \sigma_n(x)) \\
&\vdots \\
z_i &= (\sigma_1'(x), ..., \sigma_i'(x), ..., \sigma_n(x)) \\
&\vdots \\
z_n &= (\sigma_1'(x), ..., \sigma_i'(x), ..., \sigma_n'(x)),
\end{aligned}
$$

where $z_{i,j}$ denotes $\sigma_j(x) \in z_i$. Now, there exists a $Z' \subseteq Z$ such that $z_i \in Z'$ if and only if $\sigma_i(x) > \sigma_i'(x)$ for all $i \in N$. For some $z_i \in Z'$, it is obvious that $\mathcal{C}(z_i)(x) < \mathcal{C}(z_{i-1})(x)$, where z_{i-1} is not necessarily in Z'. The proof now shows that i can manipulate $\mathcal{C}$ with two cases. First suppose $\sigma_i(x) > \mathcal{C}(z_i)(x)$. Then, $\mathcal{C}(z_i)(x) < \mathcal{C}(z_{i\setminus i}, z_{i-1,i})$, then i can manipulate $\mathcal{C}$ at z_i by submitting $z_{i-1,i}$ rather than $z_{i,i}$. Second suppose $\sigma_i(x) \leq \mathcal{C}(z_i)(x)$. Then by assumption, $\sigma_i(x) < \mathcal{C}(z_{i-1})(x)$, and $\mathcal{C}(z_{i-1})(x) > \mathcal{C}(z_{i-1\setminus i}, z_{i,i})$. Thus, i can manipulate $\mathcal{C}$ at z_{i-1} by submitting $z_{i,i}$ rather than $z_{i-1,i}$. □

Proposition 11 demonstrates that strategy-proofness is sufficient for a monotonicity. However, previous research in crisp preference relations has shown that strategy-proofness is necessary and sufficient [27]. This is does not hold in the fuzzy framework as the following corollary demonstrates.

Corollary 14. *Montonicity does not imply strategy-proofness.*

Proof. A counter example will suffice. For any $x \in X$ and any $\sigma \in \mathcal{F}(X)^n$, let $\mathcal{C}(\sigma)(x) = \left(\frac{1}{n}\right) \sum_{\forall i \in N} \sigma_i(x)$. It is easy to verify that $\mathcal{C}$ is a monotonic choice function. Now let $x \in X$ and suppose $N = \{1, 2, 3\}$ and $\sigma(x) = (.4, .1, .6)$. In this case, $\mathcal{C}(\sigma)(x) = \frac{1}{3}(.4 + .1 + .6) = .367$. Obviously, some $i \in \{1, 3\}$ could manipulate $\mathcal{C}$ with some $\sigma_i'(x) > \sigma_i(x)$. □

The following defintion is necessary to characterize the domain of strategy-proof choice functions. The proceeding argument applies the same logic as Moulin [26].

Definition 15. (*Augmented median rule*). Let $M : \mathcal{F}(X)^n \to \mathcal{F}(X)$ be a fuzzy choice function defined as follows:

$$M(\sigma)(x) = \text{med}(p_1, ..., p_{n-1}, \ \sigma_1(x), ..., \sigma_n(x))$$

where $\forall p_i \in [0, 1]$.[3]

$\{p_1, ..., p_{n-1}\}$ is a set of predefined phantom alternatives that serve two purposes. First, it allows $M(\sigma)$ to be generalized to any type of rank-selecting function such as minimum or maximum. Second, the set also ensures an odd number of alternatives thereby ensuring that the a median can always be selected.

Lemma 16. *$M(\sigma)$ is a strategy-proof fuzzy choice function.*

Proof. Assume $M(\sigma)$ is defined as given above. Suppose $M(\sigma)$ is not strategy-proof. This leads to a contradiction.

There are two cases to cosider. First, let $x \in X$, $i \in N$ and $\sigma \in \mathcal{F}(X)$. Suppose $\sigma_i(x) < M(\sigma)(x)$. Then there exists a $\sigma' \in \mathcal{F}(X)$ such that $M(\sigma)(x) > M(\sigma_{N\setminus i}, \sigma'_i)$. For clarity, let $M(\sigma)(x) = a$ and $M(\sigma_{N\setminus i}, \sigma'_i) = b$. Obviously, $a \neq b$ and $a > b$.

Note that $a \in \{p_1, ..., p_{n-1}, \sigma_1(x), ..., \sigma_i(x), ..., \sigma_n(x)\}$. Because $\sigma_i(x) < a$, $\sigma'(x) \nleq \sigma(x)$, else $\text{med}(p_1, ..., p_{n-1}, \sigma_1(x), ..., \sigma_i(x), ..., \sigma_n(x)) = \text{med}(p_1, ..., p_{n-1}, \sigma_1(x), ..., \sigma'_i(x), ..., \sigma_n(x))$. Thus, $\sigma'_i(x) > \sigma(x)$. This implies then $b = \text{med}(p_1, ..., p_{n-1}, \sigma_1(x), ..., \sigma'_i(x), ..., \sigma_n(x)) \geq a$. However, $M(\sigma_{N\setminus i}, \sigma'_i) = b$ and $a > b$. This is a contradiction.

Second, suppose $\sigma_i(x) > M(\sigma)(x)$. Then there exists a $\sigma' \in \mathcal{F}(X)$ such that $M(\sigma)(x) < M(\sigma_{N\setminus i}, \sigma'_i)$. Again, let $M(\sigma)(x) = a$ and $M(\sigma_{N\setminus i}, \sigma'_i) = b$. Obviously, $a \neq b$ and $a < b$. Because $\sigma_i(x) > a$, $\sigma'(x) \ngeq \sigma(x)$, else $\text{med}(p_1, ..., p_{n-1}, \sigma_1(x), ..., \sigma_i(x), ..., \sigma_n(x)) = \text{med}(p_1, ..., p_{n-1}, \sigma_1(x), ..., \sigma'_i(x), ..., \sigma_n(x))$. Thus, $\sigma'_i(x) < \sigma(x)$. However, as before, $b =$

[3]Several studies characterize the augmented median rule using $\{p_1, ..., p_{n+1}\}$ [3, 5, 26]. In this case, however, by setting $p_1 = 0$ and $p_{n+1} = 1$, the rule can be more succinctly written using $n - 1$ alternatives.

med$(p_1, ..., p_{n-1},\, \sigma_1(x), ...,\, \sigma_i'(x), ...,\, \sigma_n(x)) \leq a$, a contradiction. The desired result now follows. □

In words, proposition 14 demonstrates that if $i \in N$ attempts to manipulate the value of $M(\sigma)(x)$ with σ_i', one of two events will happen. Either the new manipulated social choice will be identical to the original social choice or the new manipulated social choice will move further away from i's ideal social intensity for $x \in X$. Hence, $i \in N$ will not be better off by reporting any $\sigma_i' \neq \sigma_i$.

The paper's theorem that $\mathcal{C}(\sigma)$ is strategy-proof if and only if $\mathcal{C}(\sigma) = M(\sigma)$ follows the logic in Ching [9] and makes use of the following lemma.

Lemma 17. *Let $\bar{\sigma}_i(x) = 1$ and $\underline{\sigma}_i(x) = 0$ for all $x \in X$ and all $i \in N$. A fuzzy choice function $\mathcal{C}$ is strategy-proof if and only if, for all $\sigma \in \mathcal{F}(X)^n$, all $x \in X$ and all $i \in N$, the following holds:*

$$\mathcal{C}(\sigma)(x) = med\{\sigma_i(x), \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x), \mathcal{C}(\sigma_{N\setminus i}, \bar{\sigma}_i)(x)\}$$

Proof. Suppose $\mathcal{C}$ is a strategy-proof fuzzy choice function. Let $\sigma \in \mathcal{F}(X)^n$ and $x \in X$. By monotonicty and proposition 13, $\mathcal{C}(\sigma_{N\setminus i}, \bar{\sigma}_i)(x) \geq \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x)$. There are three cases to consider to prove the relationship.

First, suppose $\sigma_i(x) \in (\mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x), \mathcal{C}(\sigma_{N\setminus i}, \bar{\sigma}_i)(x))$. Further, suppose $\mathcal{C}(\sigma)(x) < \sigma_i(x)$, then i can submit $\bar{\sigma}_i(x)$ where $\mathcal{C}(\sigma_{N\setminus i}, \bar{\sigma}_i)(x) > \sigma_i(x) > \mathcal{C}(\sigma)(x)$. Thus, $\mathcal{C}(\sigma)(x) < \mathcal{C}(\sigma_{N\setminus i}, \bar{\sigma}_i)(x)$, and $\mathcal{C}$ is manipulable, a contradiction. Now suppose $\mathcal{C}(\sigma)(x) > \sigma_i(x)$. Similarily, $\mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x) < \sigma_i(x) < \mathcal{C}(\sigma)(x)$. Because $\mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x) < \mathcal{C}(\sigma)(x)$, $\mathcal{C}$ is manipulable, a contradiction. Hence $\mathcal{C}(\sigma)(x) = \sigma_i(x)$ when $\sigma_i(x) \in (\mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x), \mathcal{C}(\sigma_{N\setminus i}, \bar{\sigma}_i)(x))$.

Second, suppose $\sigma_i(x) \leq \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x)$. To see that $\mathcal{C}(\sigma)(x) = \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x)$, suppose $\mathcal{C}(\sigma)(x) < \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x)$. Then $\underline{\sigma}_i(x) \leq \mathcal{C}(\sigma)(x) < \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x)$. In this case, i can manipulate $\mathcal{C}$ at $(\sigma_{N\setminus i}, \underline{\sigma}_i)$ by submitting σ_i rather than $\underline{\sigma}_i$, a contradition. Likewise, suppose $\mathcal{C}(\sigma)(x) > \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x) \geq \sigma_i(x)$. Now, i can manipulate $\mathcal{C}$ at σ by submitting $\underline{\sigma}_i$ rather than σ_i. Hence, $\mathcal{C}(\sigma)(x) = \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x)$ when $\sigma_i(x) \leq \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x)$.

Third, suppose $\sigma_i(x) \geq \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x)$ and $\mathcal{C}(\sigma)(x) > \mathcal{C}(\sigma_{N\setminus i}, \overline{\sigma}_i)(x)$. Then $\bar{\sigma}_i(x) \geq \mathcal{C}(\sigma)(x) > \mathcal{C}(\sigma_{N\setminus i}, \overline{\sigma}_i)(x)$, where i can manipulate $\mathcal{C}$ at $(\sigma_{N\setminus i}, \overline{\sigma}_i)$ by submitting σ_i rather than $\bar{\sigma}_i$. Now suppose $\mathcal{C}(\sigma)(x) < \mathcal{C}(\sigma_{N\setminus i}, \overline{\sigma}_i)(x)$. Then $\mathcal{C}(\sigma)(x) < \mathcal{C}(\sigma_{N\setminus i}, \overline{\sigma}_i)(x) \leq \sigma_i(x)$, and i can manipulate $\mathcal{C}$ at σ by submitting $\bar{\sigma}_i$ rather than σ_i. Hence, $\mathcal{C}(\sigma)(x) = \mathcal{C}(\sigma_{N\setminus i}, \bar{\sigma}_i)(x)$ when $\sigma_i(x) \geq \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x)$.

The preceding arguments prove that if a fuzzy choice function $\mathcal{C}$ is strategy-proof, then $\mathcal{C}(\sigma)(x) = \text{med}\{\sigma_i(x), \mathcal{C}(\sigma_{N\setminus i}, \underline{\sigma}_i)(x), \mathcal{C}(\sigma_{N\setminus i}, \bar{\sigma}_i)(x)\}$. Sufficiency is easily obtained from a argument similar to the one in lemma 16. □

Theorem 18. *Any fuzzy collective choice function $\mathcal{C}$ is strategy-proof if and only if it is a fuzzy augmented median voter rule.*

Proof. Once we have established that strategy-proofness implies σ-only (proposition 10) and the relationship in lemma 17, necessity follows from Ching [9]. Sufficieny follows from lemma 16. □

5 Implications for the Spatial Model

Lemma 17 and theorem 18 illustrate that a fuzzy choice function is strategy-proof if and only if it is a form of the fuzzy augmented median rule from definition 15. Further, in contrast to previous results using traditional preference relations, this relationship holds without restricting the domain of individaul preferences, $\mathcal{F}(X)^n$. While the representation of an individual's preferences with the σ_i function clearly produces a transitive preference relations R_i, where $xR_iy \iff \sigma_i(x) \geq \sigma_i(y)$, the use of the σ_i creates substantive differences between the structure of traditional and fuzzy strategy-proof choice functions The reason these differences emerge is the group of individuals are no longer deciding what alternative to select but rather deciding the degree to which the group chooses each alternative.

To illustrate the these difference, the model presented in section 3 can be applied to the spatial model, where the set of alternatives X becomes of subset of k-dimensional Euclidean space or $\mathbb{R}^k$. When $k = 1$, σ_i can be presented by a traditional fuzzy number, i.e. $\sigma_i : \mathbb{R}^1 \rightarrow [0, 1]$, which has a similar defintion to a fuzzy subset. Further, it is often assumed that σ_i is *normal*, which requires that there exists $x \in X$ such that $\sigma_i(x) = 1$. In words, normality ensures that every actor views at least one alternative as ideal. While the condition seems innocuous and strongly related to the standard assumptions of spatial models, it is not necessary.

Figure 1 illustrates an three player fuzzy preference profile where each σ_i is represented by a normal fuzzy number in one-dimensional space. It is obvious that the fuzzy number representation allow for greater variation in individual preferences that a traditional single-peaked profile. In this example, not only are the fuzzy preferences able to capute the single-plateau characteristics of concern to some scholars [7, 10, 25], but they also allow for non-single-peaked preferences (player 2), which is one substantive difference between exact and fuzzy choice. Further, the shaded areas show the social choice induced by the fuzzy median rule. To see that the social choice is indeed strategy proof even with non-single-peaked preferences, consider $x_1 \in X$. Here, $\sigma_1(x_1) > 0$, $\sigma_2(x_1) = 0$, and $\sigma_3(x_1) = 0$. Regardless of any $\sigma_1' \in \mathcal{F}(X)$ and any possible values of $\sigma_1'(x_1)$, $M(\sigma)(x_1) = 0$. In addition, consider $x_2 \in X$, where $\sigma_1(x_2) < M(\sigma)(x_2)$, $\sigma_2(x_2) > M(\sigma)(x_2)$, and $\sigma_3(x_2) = M(\sigma)(x_2)$. Similarily, player 1 cannot manipulate the fuzzy choice for x_2. For any $\sigma_1' \in \mathcal{F}(X)$ and any specific value of $\sigma_1'(x_2)$, $M(\sigma_{N\setminus 1}, \sigma_1')(x_2) \geq M(\sigma)(x_2)$. Thus, player 1 can only move the degree of social choice further away from her truthful preference for x_2.

When working in multidimensional space, the framework of the σ_i function remains largely the same, where $\sigma_i : \mathbb{R}^k \rightarrow [0, 1]$. When $k = 2$, we are interested in fuzzy subsets where every element in the image of σ_i except $\{0\}$, denoted $\text{Im}(\sigma_i)\setminus\{0\}$, is the interior and boundary of a simple closed curve. A simple closed curve is a curve for which there is a one-to-one continuous function of the unit circle onto it. In addition, a simple closed curve has an interiour that is bounded and an exterior, but there is no need for the curve to be convex. Finally, we can restrict σ_i in particular way such that, for all $t \in \text{Im}(\sigma_i)\setminus\{0\}$, $\{x \in X \mid \sigma_i(x) = t\}$ forms a compact set.

Figure 1: Three-player example of the fuzzy median rule in one-dimensional space

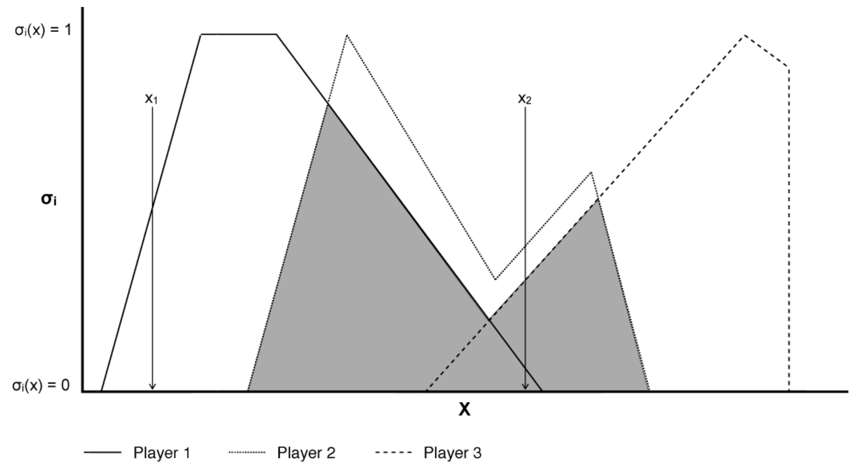
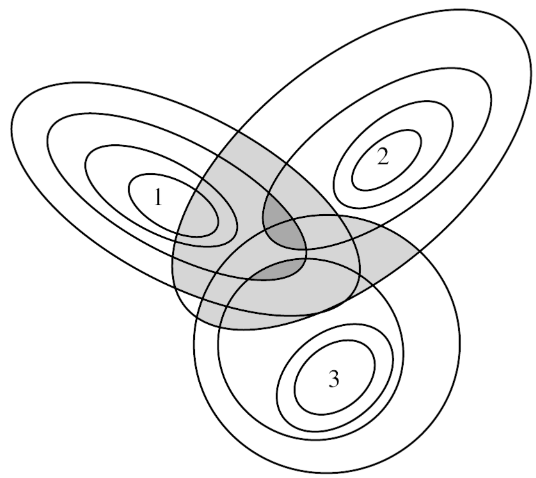


Figure 2 presents a three player fuzzy preference profile in two-dimensional space, and the σ_i function becomes a third dimension perpendicular to both the X and Y policy dimensions. In this case, $\text{Im}(\sigma_i) = \{0, .25, .5, .75, 1.0\}$, where $\sigma_i(x) = 1$ can be represented by individual i's inner-most indifference circe and $\sigma_i(x) = 0$ signifies the area outside i's outer-most indifference circle. When fuzzy preferences are constructed in this manner, they are similar to a Likert scale. As in the previous example, the shaded gray areas show the social choice induced by the fuzzy median rule, and darker areas represent a more intense social choice. Unlike the exact case, the fuzzy median rule remains strategy-proof in two-dimensional without using the dimenson-by-dimenson median rule.

Finally, another substantive difference occurs when no players have intersecting σ_i functions. When this happens, $M(\sigma)(x) = 0$ for all $x \in X$, and the group of playes rejects all possible alternatives. In this case, it is unclear as to what the social choice is. In the traditional appraoch, a choice function associates an alternative to all possible combination of individual preferences. However, in the fuzzy case when the choice function is designating a social choice intensity to each alternative, it is possible that a strategy-proof choice function assigns a zero intensity to all alternatives. This is not necessarily a misrepresentation of the

Figure 2: Fuzzy median rule in two-dimensional space



original intention of strategy-proof choice functions if rejecting all alternatives is some type of social choice.

6 Conclusion

This paper proposed a framework for charaterizing strategy-proof fuzzy choice functions in which individual preferences and the social choice are represented by fuzzy subsets of the set of alternatives. Essentially, actors are deciding to what degree the group chooses each alternative rather than choosing a specific alternative, which is the approach taken in previous studies of both exact and fuzzy social choice. Similar to previous results, strategy-proof fuzzy choice functions satisfy conditions of σ-only, weak Paretianism and monotonicity. In addition, theorem 18 demonstrates a fuzzy choice function is strategy-proof if and only if it is the fuzzy augmented median voter theorem. Unlike previous results, strategy-proof choice functions do not require any restrictions on the consistency of individual preferences or the dimensionality of the alternative space. In fact, section 5 illustrates strategy-proof fuzzy choice when individuals have multi-peaked preferences on a single dimension of alternatives and when the set of alternatives is multidimensional. The results speak to recent debates about the possibility of strategic manipulation of exact choice fucntions with single-peaked preferences. They suggest that when social choice selects alternatives to various degrees there exists strategy-proof choice functions that do not require restrictions on individuals' preferences.

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